Periodic solutions to the non-autonomous Oseen-Navier-Stokes equations in exterior domains

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Abstract: In this paper, we investigate the existence and uniqueness of periodic mild solutions to the non-autonomous Oseen-Navier-Stokes equations (ONSE) in the exterior domain $\Omega \subset \mathbb{R}^3$ of a rotating obstacle that is translating with a time-dependent velocity. Our method is based on the $L^p - L^q$ smoothness of the evolution family corresponding to linearized equations in combination with interpolation spaces and fixed-point theorems.

Keywords: Evolution families, Periodic solutions, Oseen-Navier-Stokes equations, rotating and translating obstacle.

1. Introduction and preliminaries

We consider the flow of an incompressible, viscous fluid in the exterior of a rotating obstacle that is translating with a time-dependent velocity. Here the angular velocity of the obstacle also depends on time and the axis of rotation may change. The equations describing this problem are the Oseen-Navier-Stokes equations in a time-dependent exterior domain with a prescribed velocity field at infinity. After rewriting the problem on a fixed exterior domain $\Omega \subset \mathbb{R}^3$, the system is reduced to

$$\begin{cases} u_t + (u\cdot\nabla)u - \Delta u + \nabla p = (\eta + \omega\times x)\cdot\nabla u - \omega\times u + divF & in\ \Omega\times(0,\infty), \\ \nabla\cdot u = 0 & in\ \Omega\times(0,\infty), \\ u = \eta + \omega\times x & on\ \partial\Omega\times(0,\infty), \\ u|_{t=0} = u_0 & in\ \Omega, \\ \lim_{|x|\to\infty} u = 0 & \end{cases} \qquad (1.1)$$

where[2*] $u = (u_1(x,t), u_2(x,t), u_3(x,t))^T$ is supposed to be the velocity of the fluid; $p = p(x,t)$ - the pressure; and $divF$ is the external force for a 2^{nd}-order tensor $F = F(x,t)$. Meanwhile, $\eta = (0,0,a(t))^T$ and $\omega = (0,0,k(t))^T$ stand for the translational and angular velocities respectively of the obstacle. Here $\Omega = \mathbb{R}^3 \setminus D(0)$ with $D(0)$ being the position of $\Omega \subset \mathbb{R}^3$ at $t=0$. The investigation of existence and uniqueness of a T-periodic solution to (1.1) is an important research direction related to dynamics of such evolution equations. (Serrin, 1959) in the late 60's proved an important theorem on the existence of time-periodic strong solutions to NSE in bounded domains using the stability of the solutions. (Yamazaki, 2000) used interpolation theorems and the method of Kato-iteration to investigate the periodic mild solutions on exterior domains and obtained their existence in weak-L^p spaces. (Galdi, 2022) proved the existence of the periodic solution of such equations in L^2 space using Galerkin method. Moreover, (Nguyen and Tran, 2024) used Cesaro sums and Massera methodology to obtain the existence of periodic mild solutions.

Inspired by Serrin's technique, in the present paper, we will investigate the periodic solution on weak-L^p space over Ω, i.e., we consider the existence, uniqueness of periodic mild solutions to ONSE on R+ and with data in L^p spaces. Our method is based on the $L^p - L^q$ smoothing of the evolution family $(U(t,s))_{t\geq s\geq 0}$ generated by the family of operators $L(t)$, combining with interpolation functors and ergodic method. The keys of our strategy are lying on the duality estimates, the smoothing properties and interpolation functors for the evolution family $(U(t,s))_{t\geq s\geq 0}$. Then, we can pass to the semi-linear equations using fixed point arguments. Our main results are contained in Theorems 2.2, 3.1 respectively.

Here, we recall some preliminaries on function and interpolation spaces for latter use.

Given an exterior domain Ω of class $C^{1,1}$ in $\mathbb{R}^3$, we

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denote by $C_0^\infty(\Omega)$ the space of all smooth functions with compact supports in Ω. Then, we consider the following spaces:

$$C_0^\infty(\Omega)^3 := \{(v_1, v_2, v_3) : v_j \in C_0^\infty(\Omega),\ j = 1,2,3\},$$

$$C_{0,\sigma}^\infty(\Omega)^3 := \{v \in C_0^\infty(\Omega)^3 : div v = 0 \text{ in } \Omega\},$$

$$L_\sigma^p(\Omega)^3 := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{L^p}}. \quad (1.2)$$

We note that, as in the works (T. Hansel and A. Rhandi, 2014; T. Hishida, 2020), the regularity of boundary is needed forthen well-posedness and $L^p - L^q$-smoothing property of the linearized problem, and $C^{1,1}$-regularity was enough for such properties. The Lorentz space $L^{r,q}(\Omega)^3, (1 < r < \infty, 1 \le q \le \infty)$, was defined in (J. Bergh and J. Lofstrom, 1976; H. Komatsu, 1981; H. Tribel, 1978), and here $L^{r,r}(\Omega)^3 = L^r(\Omega)^3$. Moreover, $L^{r,\infty}(\Omega)^3$ is called the weak-L^r space denoted by $L_w^r(\Omega)^3 := L^{r,\infty}(\Omega)^3$.

Denote by $\|.\|_{r,w}$ the norm in $L_w^r(\Omega)^3$. We take the following inequality from (W. Borchers and T. Miyakawa, 1995, Lemma 2.1) which is known as weak Holder inequality.

Lemma 1.1. Consider indices p, q, r satisfying $1 < p \le \infty, 1 < q < \infty$, $1 < r < \infty$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Then, $\|fg\|_{r,w} \le C\|f\|_{p,w}\|g\|_{q,w}$ for $f \in L_w^p(\Omega)^3, g \in L_w^q(\Omega)^3$, (1.3)

where C is a positive constant depending only on p and q. Note that $L_w^\infty(\Omega)^3 = L^\infty(\Omega)^3$.

Denote $\mathrm{P} = \mathrm{P}_r$ the Helmholtz projection on $L^r(\Omega)$ ($1 < r < \infty$), i.e., the projection onto $L_\sigma^r(\Omega)^3$ relative to the Leray-Helmholtz decomposition (see [W. Borchers and T. Miyakawa, 1995]):

$$L^r(\Omega)^3 = L_\sigma^r(\Omega)^3 \oplus \{\nabla p \in L^r(\Omega)^3 : p \in L_{loc}^r(\bar{\Omega})\}.$$

Next, for each $t \ge 0$ we define the operator $\mathrm{L}(t)$ as follows:

$$D(\mathrm{L}(t)) := \{u \in L_\sigma^r(\Omega)^3 \cap W_0^{1,r}(\Omega)^3 \cap W^{2,r}(\Omega)^3 : (\omega(t) \times x) \cdot \nabla u \in L^r(\Omega)^3\}$$
$$\mathrm{L}(t)u := -P[\Delta u + (\eta(t) + \omega(t) \times x) \cdot \nabla u - \omega(t) \times u] \text{ for } u \in D(\mathrm{L}(t)). \quad (1.4)$$

It is known that the family of operators $\{\mathrm{L}(t)\}_{t\ge 0}$ generates a bounded evolution family $\{U(t,s)\}_{t\ge s\ge 0}$ on $L_\sigma^r(\Omega)^3$ for each $1 < r < \infty$ under the conditions that $\eta, \omega \in C_{loc}^\theta([0,\infty); \mathrm{R}^3)$ for some $\theta \in (0,1)$ (see (T. Hansel and A. Rhandi, 2014)). Furthermore, the solenoidal Lorentz spaces are identified (see (W. Borchers and T. Miyakawa, 1995)) by

$$L_\sigma^{r,q}(\Omega)^3 := (L_\sigma^{r_1}(\Omega)^3, L_\sigma^{r_2}(\Omega)^3)_{\theta,q} \quad (1.5)$$

where $1 < r_0 < r < r_1 < \infty, 1 \le q \le \infty$ and $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. Then $\{U(t,s)\}_{t\ge s\ge 0}$ are extended to strongly continuous, bounded evolution operators on $L_\sigma^{r,q}(\Omega)^3$. Denote also by $L_{\sigma,w}^r(\Omega)^3 := L_\sigma^{r,\infty}(\Omega)^3$.

Also, for $1 \le q < \infty$, we have the dual space

$$(L_\sigma^{r,q}(\Omega)^3)' = L_\sigma^{r',q'}(\Omega)^3 \text{ here } r' = \frac{r}{r-1}, q' = \frac{q}{q-1} \text{ and } q' = \infty \text{ if } q = 1. \quad (1.6)$$

Moreover, for $0 < \theta < 1$ we consider the space of Holder continuous functions

$$C^\theta([0,\infty); \mathrm{R}^3) := \{f \in C([0,\infty); \mathrm{R}^3) : \sup_{t > s \ge 0} \frac{|f(t) - f(s)|}{(t-s)^\theta} < \infty\}.$$

We analyze the case in which both $\eta(t)$ and $\omega(t)$ are prescribed T-periodic functions such that

$$\eta, \omega \in C^\theta([0,\infty); \mathrm{R}^3) \cap C^1([0,\infty); \mathrm{R}^3) \cap L^\infty([0,\infty); \mathrm{R}^3) \text{ with some } \theta \in (0,1). \quad (1.7)$$

Let us introduce the following notations:

$$|(\eta, \omega)|_0 := \sup_{T \ge t \ge 0}(|\eta(t)| + |\omega(t)|),$$

$$|(\eta, \omega)|_1 := \sup_{T \ge t \ge 0}(|\eta'(t)| + |\omega'(t)|),$$

$$|(\eta,\omega)|_\theta := \sup_{T\ge t>s\ge 0} \frac{|\eta(t)-\eta(s)|+|\omega(t)-\omega(s)|}{(t-s)^\theta} \tag{1.8}$$

There is a constant $m \in (0,\infty)$ such that

$$|(\eta,\omega)|_0 + |(\eta,\omega)|_1 + |(\eta,\omega)|_\theta \le m \tag{1.9}$$

We recall the following $L^{r,q} - L^{p,q}$ estimates taken from (T. Hishida, 2020, Theorem 2.1, Theorem 2.2).

Proposition 1.2. Suppose that η and ω fulfill (1.7) and (1.9) for an $m \in (0,\infty)$. Denote by $\|.\|_{r,q}$ the norm in $L^{r,q}$ (here $1 < r < \infty, 1 \le q \le \infty$).

Then, $\|U(t,s)x\|_{r,q} \le M(t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \|x\|_{p,q}$ for all $t > s \ge 0$ (here $1 < p \le r < \infty$). (1.10)

We take $R_0 > 0$ satisfying

$$\mathbb{R}^3 \setminus \Omega \subset B_{R_0} := \{x \in \mathbb{R}^3; |x| < R_0\}. \tag{1.11}$$

Then, we fix a cut-off function $\phi \in C_0^\infty(B_{3R_0})$ such that $\phi = 1$ on B_{2R_0} and set

$$b(x,t) = \frac{1}{2}\mathrm{rot}\{\phi(x)(\eta(t)\times x - |x|^2 \omega(t))\}$$

which fulfills $\mathrm{div} b = 0, \quad b|_{\partial\Omega} = \eta + \omega \times x, \quad b(t) \in C_0^\infty(B_{3R_0})$.

Moreover, an elementary calculation shows that

$$\omega \times b = div\begin{pmatrix} \frac{-(a(t))^2|x|^2\phi(x)}{2} & 0 & a(t)k(t)x_2\phi(x) \\ 0 & \frac{-(a(t))^2|x|^2\phi(x)}{2} & -a(t)k(t)x_1\phi(x) \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{div}(-F_1) \tag{1.12}$$

$$b_t = \mathrm{div}\begin{pmatrix} 0 & \frac{-a'(t)|x|^2\phi(x)}{2} & \frac{-k'(t)x_1\phi(x)}{2} \\ \frac{a'(t)|x|^2\phi(x)}{2} & 0 & \frac{-k'(t)x_2\phi(x)}{2} \\ k'(t)x_1\phi(x) & k'(t)x_2\phi(x) & 0 \end{pmatrix} = \mathrm{div}(-F_2) \tag{1.13}$$

If we set $z(x,t) = u(x,t) - b(x,t)$ then the fact that u fulfills (1.1) is equivalent to z satisfies

$$\begin{cases} z_t - \Delta z - (\eta + \omega\times x)\cdot\nabla z + \omega \times z + \nabla p = divG - (z\cdot\nabla)z - (b\cdot\nabla)z - (z\cdot\nabla)b - (b\cdot\nabla)b \\ \qquad\qquad in\ \Omega\times(0,\infty), \\ \nabla\cdot z = 0 \qquad in\ \Omega\times(0,\infty), \\ z = 0 \qquad on\ \partial\Omega\times(0,\infty), \\ z|_{t=0} = z_0 \qquad in\ \Omega, \\ \lim_{|x|\to\infty} u = 0, \end{cases} \tag{1.14}$$

where $z_0(x) := u_0(x) - b(x,0)$ and

$$G := F + F_1 + F_2 + \nabla b + (\eta + \omega\times x)\otimes\nabla b. \tag{1.15}$$

Applying Helmholtz operator P to (1.14) we may rewrite the equation as a non-autonomous abstract Cauchy problem

$$\begin{cases} z_t + L(t)z = Pdiv\big(G - z\otimes z - b\otimes z - z\otimes b - b\otimes b\big) \\ \quad z|_{t=0} = z_0 \in L^3_{\sigma,w}(\Omega)^3, \end{cases} \tag{1.16}$$

where $L(t)$ is defined as in (1.4).

As proved in (T. Hansel, 2014), the family of operators $(\mathrm{L}(t))_{t\geq 0}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ in the sense, roughly speaking, that $z(t) = \mathrm{U}(t,0)z_0$ is the solution to homogeneous equation $z_t + L(t)z = 0; z(0) = z_0$. Therefore, we can define a mild solution of Equation (1.16) as the function $z(t)$ fulfilling the following integral equation in which the integral is understood in weak sense as in (M. Yamazaki, 2000, Remark 1.2):

$$z(t) = U(t,0)z(0) + \int_0^t U(t,\tau)\mathrm{Pdiv}(-z\otimes z - b\otimes z - z\otimes b - b\otimes b + G(\tau))d\tau \text{ for } t\geq 0. \tag{1.17}$$

Denote by $\mathbb{R}_+ := (0,\infty)$ and write $\|.\|_{s,w}$ for the norm in $L^s_{\sigma,w}(\Omega)^3$.

The following space is also needed in our strategy:

$$C_{w*,b}(\mathbb{R}_+, L^s_{\sigma,w}(\Omega)^3) := \{v : \mathbb{R}_+ \to L^s_{\sigma,w}(\Omega)^3 \mid v \text{ is weak* continuous and } \sup_{t\in\mathbb{R}_+} \| v(t) \|_{s,w} < \infty\} \tag{1.18}$$

endowed with the norm $\|v\|_{\infty,s,w} := \sup_{t\in\mathbb{R}_+} \|v(t)\|_{s,w}$.

Remark 1.3. Let η and ω be T-periodic functions satisfying (1.7) and (1.9). Let the external force F fulfill that F belongs to $C_{w*,b}(\mathbb{R}_+, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$ and is T-periodic. Then G is T-periodic and belonging to $C_{w*,b}(\mathbb{R}_+, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$, moreover $\|G\|_{\infty,\frac{3}{2},w} \leq \|F\|_{\infty,\frac{3}{2},w} + Cm$ (1.19)

We know that $\mathrm{L}(t)$ is T-periodic, i.e., $\mathrm{L}(t+T) = \mathrm{L}(t)$ for a fixed constant $T > 0$ and all $t\in\mathbb{R}_+$. Therefore, the corresponding evolution family $(U(t,s))_{t\geq s\geq 0}$ becomes T-periodic in the sense that $U(t+T, s+T) = U(t,s)$ for all $t\geq s\geq 0$. (1.20)

We rewrite Equation (1.16) in the form

$$\begin{cases} z_t + L(t)z = Pdiv\big(g(z)(t)\big) \\ \quad z|_{t=0} = z_0 \in L^3_{\sigma,w}(\Omega)^3, \end{cases} \tag{1.21}$$

where $g(z) = G - z\otimes z - b\otimes z - z\otimes b - b\otimes b$.

2. Periodic solutions to the linear equation

The linearized equation of (1.21) is

$$\begin{cases} z_t + L(t)z = PdivG(t) \\ \quad z|_{t=0} = z_0 \in L^3_{\sigma,w}(\Omega)^3, \end{cases} \tag{2.1}$$

Using the evolution family $(U(t,s))_{t\geq s\geq 0}$ generated by $(\mathrm{L}(t))_{t\geq 0}$, we can defined mild solution to Equation (2.1) which is the function $z(t)$ satisfying the following equation in which the integral is understood in distribution sense as in (M. Yamazaki, 2000, Remark 1.2):

$$z(t) = U(t,0)z(0) + \int_0^t U(t,\tau)\mathrm{Pdiv}G(\tau)d\tau. \tag{2.2}$$

We have the following lemma for the boundedness of mild solutions to Equation (2.1):

Lemma 2.1. Let Ω be an exterior domain Ω in $\mathbb{R}^3$ with a $C^{1,1}$-boundary and $z_0 \in L^3_{\sigma,w}(\Omega)^3$. Suppose that $G\in C_{w*,b}(\mathbb{R}_+, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$. Then, Eq. (2.1) has a unique mild solution $z\in C_{w*,b}(\mathbb{R}_+, L^3_{\sigma,w}(\Omega)^3)$ which is represented in (2.2). Also, $\|z\|_{\infty,3,w} \leq M\|G\|_{\infty,\frac{3}{2},w}$, (2.3)

where M is positive constant which is independent of z_0, z and F.

Proof. See (Nguyen and Tran, 2024, Theorem 2.2, Theorem 2.4).

We obtain the main theorem of this section. In particular, we prove that if the external force F is periodic, then the mild solution of Equation (2.1) is also periodic.

Theorem 2.2. Consider an exterior domain Ω in $\mathbb{R}^3$ with a $C^{1,1}$-boundary and $z_0 \in L^3_{\sigma,w}(\Omega)^3$. Suppose that $G \in C_{w^*,b}(\mathbb{R}_+, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$ is T-periodic. Let η and ω be T-periodic functions fulfilling (1.7) and (1.9). Then, Equation (2.1) has one and only one periodic mild solution $\hat{z}$ in $C_{w^*,b}(\mathbb{R}_+, L^3_{\sigma,w}(\Omega)^3)$.

Proof. For $z(t)$ in (2.2), we now prove that $\{z(nT)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^3_{\sigma,w}(\Omega)^3$. Indeed, putting $w(t) = z(t+(m-n)T)$ for arbitrary fixed natural numbers $m > n \in \mathbb{N}$, using the periodicity of G we prove that w can be rewritten as $w(t) = U(t,0)z((m-n)T) + \int_0^t U(t,s)PdivG(s)ds$ for all $t \geq 0$. (2.4)

Indeed, $w(t) = z(t+(m-n)T)$

$$
\begin{aligned}
&= U(t+(m-n)T,0)z(0) + \int_0^{t+(m-n)T} U(t+(m-n)T,s)PdivG(s)ds \\
&= U(t+(m-n)T,(m-n)T)U((m-n)T,0)z(0) \\
&\quad + \int_0^{(m-n)T} U(t+(m-n)T,(m-n)T)U((m-n)T,s)PdivG(s)ds \\
&\quad + \int_{(m-n)T}^{t+(m-n)T} U(t+(m-n)T,s)PdivG(s)ds \\
&= U(t+(m-n)T,(m-n)T)\left(U((m-n)T,0)z(0) + \int_0^{(m-n)T} U((m-n)T,s)PdivG(s)ds\right) \\
&\quad + \int_{(m-n)T}^{t+(m-n)T} U(t+(m-n)T,s)PdivG(s)ds \\
&= U(t,0)z((m-n)T) + \int_0^t U(t,s)PdivG(s)ds .
\end{aligned}
$$

Therefore, (2.4) follows. Next, we derive from (1.10) the estimate

$$\|U(t,0)x\|_{3,w} \leq \varphi(t)\|x\|_{r,w}, \tag{2.5}$$

where the function $\varphi(t) := Mt^{-(\frac{3}{2r}-\frac{1}{2})}$ for all $t > 0$ and $\frac{3}{2} \leq r < 3$. Now, the relation in (2.5) yields

$$\|z(t)-w(t)\|_{3,w} = \|U(t,0)(z(0)-w(0))\|_{3,w} \leq \varphi(t)\|z(0)-w(0)\|_{r,w} \leq C\varphi(t),\ t>0.$$

Taking $t := nT$ on the above inequality we obtain $\|z(nT)-z(mT)\|_{3,w} \leq C\varphi(nT)$ for all $m > n \in \mathbb{N}$. From the fact $\lim_{t\to\infty}\varphi(t) = 0$, it follows that $\{z(nT)\}_{n\in\mathbb{N}}$ is Cauchy sequence in $L^3_{\sigma,w}(\Omega)^3$. Since $L^3_{\sigma,w}(\Omega)^3$ is a Banach space, the sequence $\{z(nT)\}_{n\in\mathbb{N}}$ is convergent in $L^3_{\sigma,w}(\Omega)^3$, and we put $z^* := \lim_{n\to\infty} z(nT) \in L^3_{\sigma,w}(\Omega)^3$.

Taking now z* as initial value, we then prove that the mild solution $\hat{z}(t) = U(t,0)z^* + \int_0^t U(t,s)PdivG(s)ds$ is T-periodic.

To do this, we put $v(t) := U(t+nT,0)z_0 + \int_0^{t+nT} U(t+nT,s)PdivG(s)ds$ for every fixed $n \in \mathbb{N}$ and all $t \geq 0$, i.e., $v(t) = z(t+nT)$ for

$$z(t) = U(t,0)z_0 + \int_0^t U(t,s)PdivG(s)ds \tag{2.6}$$

as in previous step. Again, by the periodicity of G we obtain that v satisfies $v(t) = U(t,0)z(nT) + \int_0^t U(t,s)PdivG(s)ds$ for z being defined as in (2.6).

We then have
$\|\hat{z}(T)-v(T)\|_{3,w}=\|U(T,0)(\hat{z}(0)-v(0))\|_{3,w}\leq\|U(T,0)\|\|\hat{z}(0)-v(0)\|_{3,w}$.
This means
$\|\hat{z}(T)-z((n+1)T)\|_{3,w}\leq\ \|U(T,0)\|\|z^*-z(nT)\|_{3,w}$.

Letting now $n\to\infty$ and using the fact that $\lim_{n\to\infty}z(nT)=z^*=\hat{z}(0)$ in $L^3_{\sigma,w}(\Omega)^3$ we obtain $\hat{z}(T)=\hat{z}(0)$. Therefore, $\hat{z}(t)$ is $T-$periodic.

The uniqueness of the T-periodic solution follows from (2.5). Namely, if z and v are two T-periodic solutions of Equation (2.1) with initial values z_0 and v_0, respectively, then $z(t)-v(t)=U(t,0)(z_0-v_0)$, and from the fact that $z(t)-v(t)$ is bounded it follows from (2.5) that $\|z(t)-v(t)\|_{3,w}=\|U(t,0)(z_0-v_0)\|_{3,w}\leq\varphi(t)\|z_0-v_0\|_{3,w}$.

Therefore, $\lim_{t\to\infty}\|z(t)-v(t)\|_{3,w}=0$. This, together with periodicity and continuity of z and v, follows that $z(t)=v(t)$ for all $t\in\mathbb{R}_+$.

3. Periodic solutions to to nonautonomous oseen-navier-stokes equations

We now prove the existence of the periodic mild solution to Oseen-Navier-Stokes Equations (ONSE). The following theorem contains our results on periodicity of solutions to nonautonomous ONSE.

Theorem 3.1. Consider an exterior domain Ω in $\mathbb{R}^3$ with a $C^{1,1}$-boundary and $z_0\in L^3_{\sigma,w}(\Omega)^3$. Suppose that $F\in C_{w*,b}(\mathbb{R}_+,L^{3/2}_{\sigma,w}(\Omega)^{3\times3})$ is T-periodic. Let η and ω be T-periodic functions fulfilling (1.7) and (1.9). If $\|F\|_{\infty,\frac{3}{2},w}$ and m are sufficiently small, Problem (1.16) possesses a unique T-periodic mild solution $\hat{z}$ on a small ball of $C_{w*,b}(\mathbb{R}_+,L^3_{\sigma,w}(\Omega)^3)$.

Proof. By Remark 1.3 we have that $G\in C_{w*,b}(\mathbb{R}_+,L^{3/2}_{\sigma,w}(\Omega)^{3\times3})$ is T-periodic. Consider the following closed set B^T_ρ defined by

$$\mathrm{B}^T_\rho:=\{v\in L^3_{\sigma,w}(\Omega)^3: v\ is\ T-periodic\ \ and\ \ \|v\|_{\infty,3,w}\leq\rho\}. \tag{3.1}$$

Concerning the estimates for g, we first have
$\|g(0)\|_{\infty,\frac{3}{2},w}=\|G-b\otimes b\|_{\infty,\frac{3}{2},w}\leq\|G\|_{\infty,\frac{3}{2},w}+\|b\otimes b\|_{\infty,\frac{3}{2},w}$.

It follows from the weak Holder inequality (1.3) that

$$\|b\otimes b\|_{\infty,\frac{3}{2},w}\leq C\|b\|^2_{\infty,3,w}\leq Cm^2. \tag{3.2}$$

Combining (3.2) and (1.19) we obtain

$$\|g(0)\|_{\infty,\frac{3}{2},w}\leq\|F\|_{\infty,\frac{3}{2},w}+Cm+Cm^2:=\gamma. \tag{3.3}$$

Thus, the sufficient smallness of m and $\|F\|_{\infty,\frac{3}{2},w}$ implies that γ is small enough. Again, for $v_1,v_2\in B^T_\rho$ by the weak Holder's inequality (1.3), we have that

$$\|g(v_1)-g(v_2)\|_{\infty,\frac{3}{2},w}=\|-v_1\otimes v_1+v_2\otimes v_2-b\otimes v_1-v_1\otimes b+b\otimes v_2+v_2\otimes b\|_{\infty,\frac{3}{2},w}$$
$$\leq\|-(v_1-v_2)\otimes v_1-v_2\otimes(v_1-v_2)-b\otimes(v_1-v_2)-(v_1-v_2)\otimes b\|_{\infty,\frac{3}{2},w}$$
$$\leq(2C\rho+2Cm)\|v_1-v_2\|_{\infty,3,w}. \tag{3.4}$$

This implies $\|g(v)-g(0)\|_{\infty,\frac{3}{2},w}\leq(2C\rho+2Cm)\|v\|_{\infty,3,w}$ (3.5) for $v_1 = v$, $v_2 = 0$. Combining (3.3) and (3.5) we get

$$\|g(v)\|_{\infty,\frac{3}{2},w} \le \|g(v)-g(0)\|_{\infty,\frac{3}{2},w} + \|g(0)\|_{\infty,\frac{3}{2},w} \le (2C\rho + 2Cm)\rho + \gamma \tag{3.6}$$

for $v \in B_\rho^T$. On B_ρ^T, we define the map T as follows

$$(Tv)(t) = U(t,0)z(0) + \int_0^t U(t,\tau)\mathrm{Pdiv}(g(v)(\tau))d\tau \text{ for } v \in B_\rho^T.$$

By Theorem 2.2, $z := Tv$ is the unique T-periodic mild solution to the equation

$$z_t + L(t)z = \mathrm{Pdiv}(g(v)(t)). \tag{3.7}$$

Moreover, by (2.3) , (3.6) and using the fact that $v \in B_\rho^T$ we have

$$\|z\|_{\infty,3,w} \le M\|g(v)\|_{\infty,\frac{3}{2},w} \le M(2Cm\rho + 2C\rho^2 + \gamma).$$

Therefore, if the norm $\|F\|_{\infty,\frac{3}{2},w}$, ρ and m are sufficiently small then T acts from B_ρ^T into itself.

For $v_1, v_2 \in \mathrm{B}_\rho^T$ putting $Tv_1 = z_1$ and $Tv_2 = z_2$, similarly by (2.3), (3.4) we obtain

$$\|z_1 - z_2\|_{\infty,3,w} \le M\|g(v_1) - g(v_2)\|_{\infty,\frac{3}{2},w} \le M(2C\rho + 2Cm)\|v_1 - v_2\|_{\infty,3,w}.$$

Thus, if $\|F\|_{\infty,\frac{3}{2},w}$, ρ and m are small enough then the mapping T acts from B_ρ^T into itself and is a contraction. Therefore, T has a unique fixed point, and this is the unique mild solution to Equation (1.16) in a small closed ball of $C_{w*,b}(\mathbb{R}_+, L^3_{\sigma,w}(\Omega)^3)$.

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