Existence result for inverse problems governed by generalized rayleigh-tokes equations

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Abstract: In this article, we study an inverse source problem related to the generalized Rayleigh–Stokes equations on Hilbert scales, with weak-valued nonlinearities and memory effects depending on the history of the state function. We establish a representation of the mild solution and then investigate crucial properties of the associated resolvent operators. Based on these analyses, the existence result is obtained by applying Banach's fixed-point principle. Finally, an illustrative example is provided to demonstrate the theoretical results.
Keywords: Rayleigh-Stokes equations, weak nonlinearity, inverse problems, existence results.

1. Introduction

Let Ω be bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n, n\geq 1$. We consider the following system

$$\partial_t\left(u+\eta g_{1-\alpha}*(-\Delta)^{\beta}u\right)+(-\Delta)^{\beta}u=h(t)p+f\left(t,u_{\rho}(t)\right) \text{ in } \Omega, 0<t\leq T, \quad (1.1)$$

with Dirichlet boundary condition $u=0$ on $\partial\Omega, 0\leq t\leq T$, (1.2)

and innitial value $u(0)=\zeta$, (1.3)

together with the final $u(T)=\xi$ (1.4)

where $\eta>0$, $g_{1-\alpha}(t)=\dfrac{t^{-\alpha}}{\Gamma(1-\alpha)}$ with $\alpha\in(0,1)$, the operator $(-\Delta)^{\beta}$ is the Laplace operator with the power $0<\beta\leq 1$ and h is a continuous function with $\int_0^T h(t)dt>0$. In (1.1), $\rho:\mathrm{R}^+\to\mathrm{R}^+$ is a continuous function such that $0\leq\rho(t)<t$ and $u_{\rho}(t)=u(\rho(t))$.

The equation (1.1) with $\beta=1$ was first introduced in (Fetecau et al., 2009). Then, the fractional Rayleigh–Stokes equation has attracted significant attention, and the equation has been the subject of many studies; see (Bazhlekova et al., 2015; Chen et al., 2013; Khan, 2009). In recent years, several works have been devoted to the direct problem in the case $p=0$ under various settings; see, for instance, (Lan and Tuan, 2022), and (Loi and Tuan, 2023). These studies address fundamental questions concerning the solvability, stability, and regularity of solutions. In particular, in (Dac et al., 2025), the authors introduced equation (1.1), analyzed the forward problem within the framework of Hilbert scales, and obtained significant results on the stability and regularity of mild solutions. Our goal is the inverse problem related to (1.1) in Hilbert scales, where the parameter p is determined by using condition (1.4). The problem of parameter identification has attracted the interest of many researchers with different types of differential equations (see (Janno and Kasemets, 2017); (Tuan et al., 2020)). In particular, the identification problem for nonlocal differential equations under $L^2(\Omega)$ settings has frequently been published (see (Tuan, 2023), (Ke et al., 2022)). In this paper, we consider (1.1)–(1.4) in Hilbert scales, which allows the nonlinearity to take values in weak regularity spaces. Noting that our results have not been addressed in (Ke et al., 2022), thanks to another way to estimate the resolvent operator. By the estimate for resolvent operators, our goal is to determine sufficient conditions that guarantee the existence of (p,u) satisfying (1.1)-(1.4).

This paper is organized as follows. The next section recalls some notations and basic facts on Hilbert scales, relaxation functions, and find a formula for the mild solution of system (1.1)-(1.4). In Section 3, we first prove several properties of the resolvent operators and then establish existence results on compact intervals, the section concludes an application of the obtained results.

2. Preliminaries

In this section, we recall some facts and notations related to Hilbert scales and relaxation functions, derive a representation of the solution to the considered problem, and provide estimates for the operators appearing in the solution formulas. Throughout this work, we use the notations $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and the standard norm on on $L^2(\Omega)$.

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From the initial setting, we introduce a sequence $\{e_n\}_{n=1}^{\infty}$ that forms an orthonormal basis of the Hilbert space $L^2(\Omega)$ and satisfying $-\Delta e_n = \lambda_n e_n$ in Ω and $e_n = 0$ on the boundary $\partial\Omega$, $n \in \mathrm{N}$. Each element e_n is an eigenfunction of the Laplacian operator with the corresponding eigenvalues λ_n. The sequence $\{\lambda_n\}_{n=1}^{\infty}$ is nondecreasing such that $\lambda_1 > 0$ and $\lambda_n \to \infty$. Then, the fractional Laplace operator can be defined as follows: $(-\Delta)^\gamma v = \sum_{n=1}^{\infty} \lambda_n^\gamma \langle v, e_n \rangle e_n$ with domain $D((-\Delta)^\gamma) = \{v \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\gamma} \langle v, e_n \rangle^2 < \infty\}$ where $\gamma \geq 0$. For $\sigma \in \mathrm{R}$, we define H^σ by $\mathrm{H}^\sigma = \left\{ v = \sum_{n=1}^{\infty} v_n e_n : \sum_{n=1}^{\infty} \lambda_n^\sigma v_n^2 < 0 \right\}$, then H^σ is a Hilbert space endowed with the product $\langle u, v \rangle = \sum_{n=1}^{\infty} \lambda_n^\sigma u_n v_n$ and the norm determined by $\| v \|_\sigma^2 = \sum_{n=1}^{\infty} \lambda_n^\sigma v_n^2$. It is obvious that $\mathrm{H}^0 = L^2(\Omega)$ with $\|\cdot\|_0 \equiv \|\cdot\|$, and these Hilbert spaces satisfy that H^{σ_1} is continuous embedding into H^{σ_2} whenever $\sigma_1 \geq \sigma_2$, then H^σ is said to be Hilbert scales of $L^2(\Omega)$.

In order to find a presentation for solution of the considered problem, we consider the following homogeneous equation:

$\omega'(t) + \eta\lambda\left(g_{1-\alpha} * \omega\right)' + \lambda\omega(t) = 0, t > 0$ and

$\omega(0) = 1$ (R1)

with inhomogeneous equation:

$v'(t) + \eta\lambda\left(g_{1-\alpha} * v\right)' + \lambda v(t) = G(t), t > 0$ and

$v(0) = v_0$. (R2)

We recall some crucial properties related to these equations, see (Dac et al., 2025), putting

$\ell(t) = 1 + \eta g_{1-\alpha}(t)$

then ℓ is a completely monotonic. As a result, we have (see (Cl´ement and Nohel, 1981))

(PC) there exists a nonincreasing and nonnegative function $k \in L^1_{loc}\left(\mathrm{R}^+\right)$ such that

$k * \ell = 1 \text{ on } (0, \infty)$.

Proposition 2.1. Suppose that η and λ are positive numbers. Then

(i) (R1) has a unique solution, denoted by $\omega(t,\lambda)$, which is nonnegative and nonincreasing on R^+. In addition, we have

$$\frac{1}{1 + \lambda k(t)^{-1}} \leq \omega(t,\lambda) \leq \frac{1}{1 + \lambda \int_0^t \left(1 + \eta g_{1-\alpha}(\tau)\right) d\tau} \quad (2.1)$$

Consequently, $\lim_{t \to \infty} \omega(t,\lambda) = 0$.

(ii) The following estimate holds $\int_0^t \omega(\tau,\lambda) d\tau \leq \lambda^{-1}\left(1 - \omega(t,\lambda)\right), \forall t > 0, \lambda > 0$. (2.2)

(iii) For each $t > 0$, the function $\lambda \mapsto \omega(t,\lambda)$ is nonincreasing.

(iv) The solution of (R2) given by $v(t) = \omega(t,\lambda) v_0 + \int_0^t \omega(t-\tau) G(\tau) d\tau$.

Proof. From Lemma 6.1 in (Vergara and Zacher, 2015), we get $\frac{1}{1 + \lambda k(t)^{-1}} \leq \omega(t,\lambda)$. The other results are demonstrated in (Dac et al., 2025). The proof is complete.

We now find a solution formula for our problem. Let

$$u_n(t) = \langle u(t), e_n \rangle \text{ and } G_n(t) = \left\langle h(t)p + f\left(t, u_\rho(t)\right), e_n \right\rangle.$$

Then, substituting into (1.1) - (1.3) gives us $u_n'(t) + \eta\lambda_n^\beta \left(g_{1-\alpha} * u_n(t)\right)' + \lambda_n^\beta u_n(t) = G_n(t)$ and $u_n(0) = \zeta_n = \langle \zeta, e_n \rangle$.

It follows from Proposition (iv) that $u_n(t)=\omega(t,\lambda_n^\beta)u_n(0)+\int_0^t \omega(t-\tau,\lambda_n^\beta)G_n(\tau)d\tau$.

Thus $u(t)=\sum_{n=1}^{\infty}\omega(t,\lambda_n^\beta)u_n(0)e_n+\sum_{n=1}^{\infty}\left(\int_0^t \omega(t-\tau,\lambda_n^\beta)G_n(\tau)d\tau\right)e_n$.

We define: $\mathrm{S}_\alpha(t)=\sum_{n=1}^{\infty}\omega(t,\lambda_n^\beta)\langle\cdot,e_n\rangle e_n$. (2.3)

that is said to be *resolvent operators*. Hence, we get a formula for solution of forward problem as follows:

$$u(t)=\mathrm{S}_\alpha(t)\zeta+\int_0^t \mathrm{S}_\alpha(t-\tau)\left[h(\tau)p+f\left(\tau,u_\rho(\tau)\right)\right]d\tau. \quad (2.4)$$

Employing (1.4), we now find a presentation for parameter p. For $t=T$, the above equation and (1.4) imply that $\xi_n=\omega(T,\lambda_n^\beta)\zeta_n+\int_0^T \omega(T-\tau,\lambda_n^\beta)\left[h(\tau)p_n+f_n\left(\tau,u_\rho(\tau)\right)\right]d\tau$,

where $\xi_n=\langle\xi,e_n\rangle,\zeta_n=\langle\zeta,e_n\rangle$ and $f_n\left(\tau,u_\rho(\tau)\right)=\left\langle f\left(\tau,u_\rho(\tau)\right),e_n\right\rangle$. Thanks to **(PC)**, we get $\int_0^T \omega(T-\tau,\lambda_n)h(\tau)d\tau\ge\int_0^T\frac{1}{1+\lambda_n^\beta k(\tau)^{-1}}h(T-\tau)d\tau\ge\frac{1}{1+\lambda_1^\beta k(T)^{-1}}\int_0^T h(\tau)d\tau>0$. Since $\int_0^T h(\tau)d\tau>0$,

we obtain $p_n=\left[\int_0^T \omega(T-\tau,\lambda_n)h(\tau)d\tau\right]^{-1}\left(\xi_n-\omega(T,\lambda_n^\beta)\zeta_n-\int_0^T \omega(T-\tau,\lambda_n^\beta)f_n\left(\tau,u_\rho(\tau)\right)d\tau\right)$.

It deduces that $p=\Re\left(\xi-\mathrm{S}_\alpha(T)\zeta-\int_0^T \mathrm{S}_\alpha(T-\tau)f\left(\tau,u_\rho(\tau)\right)d\tau\right)$ (2.5)

here $\Re$ is defined by $\Re(v)=\sum_{n=1}^{\infty}\left(\int_0^T \omega(T-\tau,\lambda_n^\beta)h(\tau)d\tau\right)^{-1}\langle v,e_n\rangle e_n$ for $v\in\mathrm{H}^\sigma$. (2.6)

The remainder of this section is devoted to establishing fundamental properties of the two key operators S_α and $\Re$, which will be crucial in the subsequent analysis.

Proposition 2.2. Let $\mathrm{S}_\alpha(t)$ and $\Re$ be given in (2.3) and (2.6), respectively. For $v\in\mathrm{H}^\sigma$, $\sigma\in\mathrm{R}$ and $T>0$, we have

(i) $\mathrm{S}_\alpha(\cdot)\in C\left([0,T];\mathrm{H}^\sigma\right)$ and $\|\mathrm{S}_\alpha(t)v\|\le\omega(t,\lambda_1)\|v\|_\sigma$, $\forall t\ge 0$. (2.7)

(ii) For $f\in C\left([0,T];\mathrm{H}^{\sigma-2\beta}\right)$ with $\sigma\ge 0$, then

$$\|\mathrm{S}_\alpha*f(t)\|_\sigma^2\le\frac{\eta^{-1}}{(1-\alpha)\Gamma(1-\alpha)}\int_0^t(t-\tau)^{\alpha-1}\|f(\tau)\|_{\sigma-2\beta}^2\,d\tau,\ \forall t\ge 0 \quad (2.8)$$

(iii) $\Re(v)\in\mathrm{H}^{\sigma-2\beta}$. In addition, we obtain $\|\Re(v)\|_{\sigma-2\beta}\le h_T\|v\|_\sigma$ (2.9)

where $h_T=\left(\int_0^T h(\tau)d\tau\right)^{-1}\left(2\left(\lambda_1^{-2\beta}+k(T)^{-2}\right)\right)^{1/2}$.

Proof. The results stated in (i) and (ii) were established in [Dac et al., 2025]. We now proceed to prove (iii). By the formulation of $\Re$, we get $\|\Re(v)\|_{\sigma-2\beta}^2=\sum_{n=1}^{\infty}\lambda_n^{\sigma-2\beta}\left(\int_0^T \omega(T-\tau,\lambda_n^\beta)h(\tau)d\tau\right)^{-2}v_n^2$.

By the fact that $\int_0^T \omega(T-\tau,\lambda_n)h(\tau)p_n d\tau\ge\int_0^T\frac{1}{1+\lambda_n^\beta k(\tau)^{-1}}h(T-\tau)d\tau\ge\frac{1}{1+\lambda_n^\beta k(T)^{-1}}h_T$

here $h_T = \int_0^T h(\tau)d\tau$, one has $\| \Re(v) \|_{\sigma-2\beta}^2 \le \sum_{n=1}^{\infty} \lambda_n^{\sigma-2\beta} \frac{\left(1+\lambda_n^{\beta} k(T)^{-1}\right)^2}{h_T^2} v_n^2 \le 2\sum_{n=1}^{\infty} \lambda_n^{\sigma-2\beta} \frac{1+\lambda_n^{2\beta} k(T)^{-2}}{h_T^2} v_n^2$

$$\le 2h_T^{-2} \sum_{n=1}^{\infty} \left(\lambda_n^{-2\beta} + k(T)^{-2}\right) \lambda_n^{\sigma} v_n^2 \le 2h_T^{-2} \left(\lambda_1^{-2\beta} + k(T)^{-2}\right) \sum_{n=1}^{\infty} \lambda_n^{\sigma} v_n^2.$$

This completes the argument and establishes the desired conclusion. The proof is complete.

3. The existence results

Let us begin by giving a definition of a mild solution to the system (1.1)-(1.4.

Definition 3.1. *For* $\sigma \ge 0$ *and let* $\zeta, \xi \in \mathrm{H}^{\sigma}$. *A pair* $(p,u) \in \mathrm{H}^{\sigma-2\beta} \times C\left([0,T];\mathrm{H}^{\sigma}\right)$ *is called a mild solution to (1.1) – (1.4) iff* $p = \Re\left(\xi - \mathrm{S}_{\alpha}(T)\zeta - \int_0^T \mathrm{S}_{\alpha}(T-\tau) f\left(\tau, u_{\rho}(\tau)\right) d\tau \right)$ *and*

$$u(t) = \mathrm{S}_{\alpha}(t)\zeta + \int_0^t \mathrm{S}_{\alpha}(t-\tau)\left[h(\tau)p + f\left(\tau, u_{\rho}(\tau)\right)\right] d\tau,\ t \in [0,T].$$

In what follows, we use the standard norm of $v \in C\left([0,T];\mathrm{H}^{\sigma}\right)$ defined by $\| v \|_{\infty} = \sup_{t\in[0,T]} \| v(t) \|_{\sigma}$. For the analysis of our problem, we shall rely on the following standard assumption.

(F) The function f is a map such that $f : [0,T] \times \mathrm{H}^{\sigma} \to \mathrm{H}^{\sigma-2\beta}$ satisfying

(F1) $f(t,0) = 0$;

(F2) if $v_1, v_2 \in \mathrm{H}^{\sigma}$ and $\| v_i \|_{\sigma} \le r, i = \overline{1,2}$ then $\| f(t,v_2) - f(t,v_1) \|_{\sigma-2\beta} \le L(r) \| v_2 - v_1 \|_{\sigma}$

where L is a nonnegative function with $\limsup_{r\to 0} L(r) = L_0 < \infty$.

Theorem 3.1. Suppose **(F)** holds. If $\frac{\eta^{-1/2} T^{\alpha/2}}{\sqrt{(1-\alpha)\alpha\Gamma(1-\alpha)}} \left(h_T \, | h |_{\infty} + 1\right) L_0 < 1$ (3.1)

then there exists $\delta > 0$ and $r_* > 0$ such that (1.1) – (1.4) has a unique solution (p,u) such that $\| u(t) \|_{\sigma} \le r_*$ for all $t \in [0,T]$, provided $\| \zeta \|_{\sigma} \le \delta$ and $\| \xi \|_{\sigma} \le \delta$.

Proof. For $u \in C\left([0,T];\mathrm{H}^{\sigma}\right)$, we denote $\Phi[u](t) = \mathrm{S}_{\alpha}(t)\zeta + \int_0^t \mathrm{S}_{\alpha}(t-\tau)\left[h(\tau)p + f\left(\tau, u_{\rho}(\tau)\right)\right] d\tau$, $t \in [0,T]$, where $p = \Re\left(\xi - \mathrm{S}_{\alpha}(T)\zeta - \int_0^T \mathrm{S}_{\alpha}(T-\tau) f\left(\tau, u_{\rho}(\tau)\right) d\tau \right)$ and let $\mathbf{B}_r$ be the closed ball in $C\left([0,T];\mathrm{H}^{\sigma}\right)$ with center at the origin and radius *r*. From Proposition 2.2 (i)-(ii), it follows that $\mathrm{S}_{\alpha}(\cdot)\zeta \in C\left([0,T];\mathrm{H}^{\sigma}\right)$ and $\mathrm{S}_{\alpha} * f(\cdot, u_{\rho})(\cdot) \in C\left([0,T];\mathrm{H}^{\sigma}\right)$. We consider the middle term $M(t) = \int_0^t \mathrm{S}_{\alpha}(t-\tau) h(\tau) p d\tau$, one gets

$$\| M(t) \|_{\sigma}^2 = \sum_{n=1}^{\infty} \lambda_n^{\sigma} \left(\int_0^t \omega(t-\tau, \lambda_n^{\beta}) h(\tau) p_n d\tau \right)^2 \le | h |_{\infty}^2 \sum_{n=1}^{\infty} \lambda_n^{\sigma} p_n^2 \left(\int_0^t \omega(t-\tau, \lambda_n^{\beta}) d\tau \right)^2$$

$\le | h |_{\infty}^2 \sum_{n=1}^{\infty} \lambda_n^{\sigma} \lambda_n^{-2\beta} p_n^2 \le | h |_{\infty}^2 \| p \|_{\sigma}^2$, here $| h | = \lim_{t\in[0,T]} | h(t) |$. Then $\| M(t) \|_{\sigma} \le | h |_{\infty} \| p \|_{\sigma}$ (3.2)

It deduces that $M \in C\left([0,T];\mathrm{H}^{\sigma}\right)$. Hence $\Phi[u] \in C\left([0,T];\mathrm{H}^{\sigma}\right)$. For $u \in \mathbf{B}_r$ with $r > 0$, we obtain

$$\| \Phi[u](t) \|_{\sigma} \le \| \mathrm{S}_{\alpha}(t)\zeta \|_{\sigma} + \| \mathrm{S}_{\alpha} * (hp)(t) \|_{\sigma} + \| \mathrm{S}_{\alpha} * f(\cdot, u_{\rho})(t) \|_{\sigma}$$

$$\leq \omega(t,\lambda_1^\beta)\|\zeta\|_\sigma + |h|_\infty \|p\|_{\sigma-2\beta} + \frac{\eta^{-1/2}}{(1-\alpha)^{1/2}\Gamma^{1/2}(1-\alpha)}\left(\int_0^t (t-\tau)^{\alpha-1}\|f(\tau,u_\rho(\tau))\|_{\sigma-2\beta}^2 d\tau\right)^{\frac{1}{2}}$$

thanks to (2.7), (2.8) and (3.2). Since $\omega(t,\lambda_1^\beta)\leq 1$, then: $\omega(t,\lambda_1^\beta)\|\zeta\|_\sigma \leq \|\zeta\|_\sigma$. (3.3)

For $t\in[0,T]$, employing **(F)** gives us

$$\int_0^t (t-\tau)^{\alpha-1}\|f(\tau,u_\rho(\tau))\|_{\sigma-2\beta}^2 d\tau \leq L(r)^2\int_0^t (t-\tau)^{\alpha-1}\|u_\rho(\tau)\|_\sigma^2 d\tau \leq L(r)^2 r^2\int_0^t \tau^{\alpha-1}d\tau \leq \frac{1}{\alpha}L(r)^2 r^2 T^\alpha$$

Consequently, we have $I(t)=\left(\int_0^t (t-\tau)^{\alpha-1}\|f(\tau,u_\rho(\tau))\|_{\sigma-2\beta}^2 d\tau\right)^{\frac{1}{2}} \leq \frac{T^{\alpha/2}}{\alpha^{1/2}}L(r)r$. (3.4)

Combining **Proposition 2.2**(iii) and (3.4) yields

$$\|p\|_{\sigma-2\beta} \leq h_T \|\xi - S_\alpha(T)\zeta - \int_0^T S_\alpha(T-\tau)f(\tau,u_\rho(\tau))d\tau\|_\sigma$$

$$\leq h_T\left(\|\xi\|_\sigma + \|\zeta\|_\sigma + \frac{\eta^{-1/2}T^{\alpha/2}}{(1-\alpha)^{1/2}\alpha^{1/2}\Gamma^{1/2}(1-\alpha)}L(r)r\right).$$

It implies that $|h|_\infty\|p\|_{\sigma-2\beta} \leq h_T\, h|_\infty\left(\|\xi\|_\sigma + \|\zeta\|_\sigma\right) + \frac{\eta^{-1/2}T^{\alpha/2}h_T|h|_\infty}{\sqrt{(1-\alpha)\alpha\Gamma(1-\alpha)}}L(r)r$ (3.5)

From (3.3), (3.4) and (3.5), we have

$$\|\Phi[u](t)\|_\sigma \leq \|\zeta\|_\sigma + h_{T.}|h|_\infty\left(\|\xi\|_\sigma + \|\zeta\|_\sigma\right) + \frac{\eta^{-1/2}T^{\alpha/2}h_T|h|_\infty}{\sqrt{(1-\alpha)\alpha\Gamma(1-\alpha)}}L(r)r + \frac{\eta^{-1/2}T^{\alpha/2}}{\sqrt{(1-\alpha)\alpha\Gamma(1-\alpha)}}L(r)r$$

$$\leq \left(1+h_{T.}|h|_\infty\right)\|\zeta\|_\sigma + h_{T.}|h|_\infty\|\xi\|_\sigma + \frac{\eta^{-1/2}T^{\alpha/2}}{\sqrt{(1-\alpha)\alpha\Gamma(1-\alpha)}}\left(h_T|h|_\infty + 1\right)L(r)r.$$

By (3.1), we can choose $\varepsilon>0$ such that $\kappa(\varepsilon)=\frac{\eta^{-1/2}T^{\alpha/2}}{\sqrt{(1-\alpha)\alpha\Gamma(1-\alpha)}}\left(\overline{h}_T|h|_\infty + 1\right)(L_0+\varepsilon)<1$ (3.6)

one can find $r_*>0$ satisfying $L(r)\leq L_0+\varepsilon$ for all $0<r\leq r_*$.

Let $\delta=\frac{1-\kappa(\varepsilon)}{2\left(1+h_T|h|_\infty\right)}$, we have $\delta>0$ due to the fact that $\kappa(\varepsilon)<1$. Moreover, if $\|\zeta\|_\sigma\leq\delta$ and $\|\xi\|_\sigma\leq\delta$ then $\|\Phi[u](t)\|_\sigma \leq \frac{1-\kappa(\varepsilon)}{2}r_* + \frac{1-\kappa(\varepsilon)}{2}r_* + \kappa(\varepsilon)r_* \leq r_*, \forall t\in[0,T]$.

It follows that $\Phi\left[\mathbf{B}_{r_*}\right]\subset\mathbf{B}_{r_*}$. We now proceed to prove that Φ is a contraction mapping on $\mathbf{B}_{r_*}$. Take $u,v\in\mathbf{B}_{r_*}$ and denote $p_u=\Re\left(\xi - S_\alpha(T)\zeta - \int_0^T S_\alpha(T-\tau)f(\tau,u_\rho(\tau))d\tau\right)$,

$$p_v=\Re\left(\xi - S_\alpha(T)\zeta - \int_0^T S_\alpha(T-\tau)f(\tau,v_\rho(\tau))d\tau\right).$$

Then $p_u - p_v = \Re\left(\int_0^T S_\alpha(T-\tau)f(\tau,v_\rho(\tau))d\tau - \int_0^T S_\alpha(T-\tau)f(\tau,u_\rho(\tau))d\tau\right)$.

Let $J(t)=\int_0^t \mathrm{S}_\alpha(t-\tau)\left[f\left(\tau,u_\rho(\tau)\right)-f\left(\tau,v_\rho(\tau)\right)\right]d\tau$. For $t\in[0,T]$, observe that

$$\|J(t)\|_\sigma^2\le \frac{\eta^{-1}}{(1-\alpha)\Gamma(1-\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|f\left(\tau,u_\rho(\tau)\right)-f\left(\tau,v_\rho(\tau)\right)\|_{\sigma-2\beta}^2\,d\tau$$

$\le \dfrac{\eta^{-1}\left(L_0+\varepsilon\right)^2T^\alpha}{(1-\alpha)\alpha\Gamma(1-\alpha)}\|u-v\|_\infty^2$, thanks to **(F)** and **Proposition 2.2**(ii). Then

$\|p_u-p_v\|_{\sigma-2\beta}^2\le \dfrac{\eta^{-1}\left(L_0+\varepsilon\right)^2T^\alpha}{(1-\alpha)\alpha\Gamma(1-\alpha)}h_T{}^2\|u-v\|_\infty^2$. Hence, for $t\in[0,T]$, we get

$$\|\Phi[u](t)-\Phi[v](t)\|_\sigma\le|h|_\infty\|p_u-p_v\|_{\sigma-2\beta}+\|J(t)\|_\sigma$$

$$\le|h|_\infty\frac{\eta^{-1/2}\left(L_0+\varepsilon\right)T^{\alpha/2}}{\sqrt{(1-\alpha)\alpha\Gamma(1-\alpha)}}h_T|u-v\|_\infty+\frac{\eta^{-1/2}\left(L_0+\varepsilon\right)T^{\alpha/2}}{\sqrt{(1-\alpha)\alpha\Gamma(1-\alpha)}}\|u-v\|_\infty$$

$$\le\frac{\eta^{-1/2}T^{\alpha/2}}{\sqrt{(1-\alpha)\alpha\Gamma(1-\alpha)}}\left(|h|_\infty h_T\text{-}1\right)\left(L_0+\varepsilon\right)\|u-v\|_\infty\le\kappa(\varepsilon)\|u-v\|_\infty .$$

Since $\kappa(\varepsilon)<1$, Φ is a contraction mapping on $\mathbf{B}_{r_*}$. The proof is complete.

Let us finish our work with a concrete example as follows: For $n=3$ and $\Omega\subset\mathrm{R}^3$, we define $f\left(t,v\right)=F(t,\|\nabla v\|)|v|^p$, and $\rho(t)=at, a\in(0,1)$, where $F:¡^+\times¡^+\to¡$ satisfying

(**f1**): $F(t,0)=0$;

(**f2**) : there exists a constant $L_F>0$ such that $|F(t,s_2)-F(t,s_1)|\le L_F|s_2-s_1|,\forall t\ge0, s_1,s_2\in¡^+$,

and $1<p\le5$. In addtion, $\|\cdot\|$ stands for the norm in $L^2(\Omega)$. In this example, the fucntion F describes a multiplicative perturbation, that depends on the energy of system. Hence, the nonlinearity function reads as $f\left(t,u_\rho(t)\right)(x)=F(t,\|\nabla u(at,x)\|)|u(at,x)|^q, t\in[0,T], x\in\Omega$.

Take $\sigma=\beta=1$, then $(p,u)\in\mathrm{H}^{-1}\times C\left([0,T];\mathrm{H}^1\right)$ and $f:[0,T]\times\mathrm{H}^1\to\mathrm{H}^{-1}$.

In what follows, we write $A\prec B$ if and only if there is some $c>0$ such that $A\le cB$. We first note that $\mathrm{H}^1=H_0^1(\Omega)$ and $\mathrm{H}^{-1}=\left(H_0^1(\Omega)\right)^*=H^{-1}(\Omega)$. For $v\in\mathrm{H}^1$, we now show that $f(t,v)\in\mathrm{H}^{-1}$. Indeed, using (**f1**) and (**f2**), one has $\|f(t,v)\|_{-1}=\|F(t,\|\nabla v\|)|v|^p\|_{-1}=|F(t,\|v\|_1)|\cdot\||v|^p\|_{-1}$

$$\prec L_F\|v\|_1\cdot\||v|^p\|_{L^{6/5}(\Omega)}=L_F\|v\|_1\cdot\left(\int_\Omega\left(|v|^p\right)^{6/5}dx\right)^{5/6}$$

$$=L_F\|v\|_1\cdot\|v\|_{L^{6p/5}(\Omega)}^p,$$

thanks to fact that $\|\nabla v\|=\|v\|_1$ and $\mathrm{H}^{-1}\supset L^{6/5}(\Omega)$ (see [Dac et al.,2023]). On the other hand, $1<p\le5$ implies that $\frac{6}{5}p\le6$. Then, we have $L^{6p/5}(\Omega)\supset L^6(\Omega)\supset\mathrm{H}^1$ (see (Dac et al.,2023)).

Consequently, it follows that $\|f(t,v)\|_{-1}\prec\|v\|_1\cdot\|v\|_{L^{6p/5}(\Omega)}^p\prec\|v\|_1^{p+1}$.

Furthermore, if $v_1,v_2\in\mathrm{H}^1$ then $\|f(t,v_2)-f(t,v_1)\|_{-1}=\|F(t,\|\nabla v_2\|)|v_2|^p-F(t,\|\nabla v_1\|)|v_1|^p\|_{-1}$

$$\le\|F(t,\|\nabla v_2\|)|v_2|^p-F(t,\|\nabla v_1\|)|v_2|^p\|_{-1}+\|F(t,\|\nabla v_1\|)|v_2|^p-F(t,\|\nabla v_1\|)|v_1|^p\|_{-1}$$

$$\prec\|v_2-v_1\|_1\cdot\|v_2\|_1^p+\|v_1\|_1\||v_2|^p-|v_1|^p\|_{-1}. \quad (3.7)$$

Since $p>1$, we have $\left||v_2|^p-|v_1|^p\right|\le\left(|v_2|^{p-1}+|v_1|^{p-1}\right)|v_2-v_1|$. Hence

$$\||v_2|^p-|v_1|^p\|_{-1}\prec\||v_2|^p-|v_1|^p\|_{L^{6/5}(\Omega)}=\left(\int_\Omega\left(|v_2(x)|^{p-1}+|v_1(x)|^{p-1}\right)^{6/5}|v_2(x)-v_1(x)|^{6/5}dx\right)^{5/6}$$

$$\le\left(\int_\Omega\left(|v_2(x)|^{p-1}+|v_1(x)|^{p-1}\right)^{\frac{6p}{5(p-1)}}dx\right)^{\frac{5(p-1)}{6p}}\left(\int_\Omega|v_2(x)-v_1(x)|^{\frac{6p}{5}}dx\right)^{\frac{5}{6p}}$$

thanks to the Holder inequality. Therefore, we get

$$\||v_2|^p-|v_1|^p\|_{-1}\prec\left(\|v_2\|^{p-1}_{L^{6p/5}(\Omega)}+\|v_1\|^{p-1}_{L^{6p/5}(\Omega)}\right)\|v_2-v_1\|_{L^{6p/5}(\Omega)}$$

$$\prec\left(\|v_2\|_1^{p-1}+\|v_1\|_1^{p-1}\right)\|v_2-v_1\|_1 \quad (3.8)$$

We arrive at $\|f(t,v_2)-f(t,v_1)\|_{-1}\prec\|v_2-v_1\|_1\cdot\|v_2\|_1^p+\|v_1\|_1\left(\|v_2\|_1^{p-1}+\|v_1\|_1^{p-1}\right)\|v_2-v_1\|_1$

$\prec\left(\|v_2\|_1^p+\|v_1\|_1\|v_2\|_1^{p-1}+|v_1\|_1^p\right)\|v_2-v_1\|_1$. So, $L(r)=3r^p$. The condition (3.1) is verified.

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