The asymptotical stability of stationary solutions to three-dimensional kelvin-voigt equations with damping and unbounded delays

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Abstract: In this paper, we consider the three-dimensional Kelvin-Voigt equations involving unbounded delays in a bounded domain $\Omega \subset \mathbb{R}^3$. We will study the asymptotical stability of stationary solutions via the construction of Lyapunov functionals.

Keywords: Kelvin-voigt equation, stationary solutions: unbounded delays.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. In this paper, we consider the following three-dimensional Kelvin-Voigt equations with delays in Ω,

$$\begin{cases} \partial_t(u-\alpha^2\nabla u)-\nu\Delta u+(u\cdot\nabla)u+\nabla p+\kappa|u|^{\beta-1}u = g(t,u_t)+h(t) & \text{in } (0,T)\times\Omega, \\ \operatorname{div} u = 0 & \text{in } (0,T)\times\Omega, \\ u(x,t)=0 & \text{in } (0,T)\times\partial\Omega, \\ u(\theta,x)=\phi(\theta,x), & \text{in } (-\infty,0]\times\Omega, \end{cases} \quad \#(1.1)$$

where $\nu > 0$ is the kinematic viscosity, $\alpha > 0, \kappa > 0, \beta \geq 1$ are three constants, $u = u(x,t) = (u_1, u_2, u_3)$ is the velocity field of the fluid, p is the pressure, h is a nondelayed external force field, g is another external force term and contains hereditary characteristic u_t, where u_t is the function defined on $(-\infty, 0]$ by $u_t(\theta) = u(t+\theta), \theta \in (-\infty, 0], \phi$ is the initial datum on the interval. [3*]

The case $\alpha \equiv 0$ and $g \equiv 0$ has been studied in [CJ08] by X. Cai and Q. Jiu, the equations 1.1 becomes Navier-Stokes equations with damping.

Note that the case $\kappa \equiv 0$ and $g \equiv 0$ corresponds to the classical Navier-Stokes-Voigt problem. The existence, long-time behavior and regularity of solutions to the 3D Navier-Stokes-Voigt equations without delays in bounded domains and unbounded domains satisfying the Poincare's inequality have been studied by many mathematicians (see [AT13, KL09, YZ11]). There are many results involving PDEs in fluid mechanics with delays ([CR04, MM11]) and many results about asymptotic behavior to PDEs ([NS2020, PS20]). However, all the results with finite delay (constant delays, bounded variable delay or bounded distributed delay) has been studied in the phase spaces $C([-h,0);X)$ and $L^2(-h,0;X)$ with a suitable Banach space X, or infinite distributed delay in $C_\gamma(X)$, where

$$C_\gamma(X) = \left\{\varphi \in C((-\infty,0];X): \lim_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } X\right\} (\gamma > 0)$$

is the Banach space endowed with the norm

$$\|\varphi\|_\gamma = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}\|\varphi(\theta)\|_X.$$

In this paper, following recent work [13] we continue studying the system (1.1) with unbounded variable delays in the following space

$$BCL_{-\infty}(X) = \left\{\varphi \in C((-\infty,0];X): \lim_{\theta\to-\infty} \varphi(\theta) \text{ exists in } X\right\}$$

which is a Banach space equipped with the norm

$$\|\varphi\|_{BCL_{-\infty}(X)} = \sup_{\theta\in(-\infty,0]} \|\varphi(\theta)\|_X$$

The main novelty of our paper is that we are interested in the problem with unbounded delays. The stability of stationary solutions to the 3D Kelvin-Voigt equations with damping and unbounded delays, has apparently not been studied previously. We will discuss the stability of stationary solution is shown via the construction of Lyapunov functionals.

The rest of the paper is organized as follows. In section 2, we will set up some spaces and lemmas which will be used in the later sections. Section 3 will be devoted to the asymptotical stability of stationary solutions.

2. Preliminaries

We consider the following space:

$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u = 0\}$.

Let H be the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ with the norm $|\cdot|$, and inner product $(\cdot,\cdot)$ defined by

$$(u,v) = \sum_{j=1}^{3} \int_\Omega u_j(x)v_j(x)dx \text{ for } u,v \in (L^2(\Omega))^3$$

We also denote V the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with norm $\|\cdot\|$, and the associated scalar product $((\cdot,\cdot))$ defined by

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$$((u,v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_j}\frac{\partial v_j}{\partial x_i} dx \text{ for } u,v \in (H_0^1(\Omega))^3$$

We use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle_{V,V'}$ for the dual pairing between V and V'. We recall the Stokes operator $A: V \to V'$ by $\langle Au, v\rangle = ((u,v))$. Denote by P the Helmholtz-Leray orthogonal projection in $(H_0^1(\Omega))^3$ onto the space V. Then $Au = -P\Delta u$, for all $u \in D(A) = (H^2(\Omega))^3 \cap V$. The Stokes operator A is a positive self-adjoint operator with compact inverse. Hence there exists a complete orthonormal set of eigenfunctions $\{w_j\}_{j=1}^{\infty} \subset H$ such that $Aw_j = \lambda_j w_j$ and ti

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_j \to +\infty \text{ as } t \to \infty$$

We have the following Poincaré inequalities

$$\|u\|^2 \ge \lambda_1 |u|^2, \forall u \in V, \#(2.1)$$

$$|u|^2 \ge \lambda_1 \|u\|_*^2, \forall u \in H$$

From (2.1), we have

$$|u|^2 \ge d_0(|u|^2 + \alpha^2\|u\|^2), \forall u \in V,$$

where $d_0 = \frac{\lambda_1}{1+\alpha^2\lambda_1}$. Furthermore, for $\alpha > 0$, the operator $I + \alpha^2 A$ has a compact inverse $(I + \alpha^2 A)^{-1}: D(A)' \to H$ with the following estimate

$$\|(I + \alpha^2 A)^{-1} u\| \le \alpha^{-2}\|u\|_*, \forall u \in V'.$$

We define the trilinear form b on $V \times V \times V$ by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \forall u,v,w \in V$$

and $B: V \times V \to V'$ by $\langle B(u,v), w\rangle = b(u,v,w)$. We can write $B(u,v) = P[(u\cdot\nabla)v]$. It is easy to check that if $u,v,w \in V$, then $b(u,v,w) = -b(u,w,v)$, and in particular,

$$b(u,v,v) = 0, \forall u,v \in V. \#(2.2)$$

Using Hölder's inequality and Ladyzhenskaya's inequality, we can choose the best positive constant c_0 such that

$$|b(u,v,w)| \le c_0\|u\|\|v\| \left|w|^{\frac{1}{2}}\right\| w\|^{\frac{1}{2}}, \forall u,v,w \in V. \#(2.3)$$

From (2.3) and using Poincaré's inequality (2.1), we obtain that

$$|b(u,v,w)| \le c_0\lambda_1^{-1/4}\|u\|\|v\|\|w\|, \forall u,v,w \in V \#(2.4)$$

We will assume that $f \in L^2(0,T;V')$. For the term g, we assume that $g : [0,T] \times BCL_{-\infty}(H) \to (L^2(\Omega))^3$, then

(g1) For any $\xi \in BCL_{-\infty}(H)$, the mapping $[0,T] \ni t \mapsto g(t,\xi) \in (L^2(\Omega))^3$ is measurable,

(g2) $g(\cdot,0) = 0$.

(g3) There exists a constant $L_g > 0$ such that, for any $t \in [0,T]$ and all $\xi,\eta \in BCL_{-\infty}(H)$,

$$|g(t,\xi) - g(t,\eta)| \le L_g\|\xi - \eta\|_{BCL_{-\infty}(H)}.$$

Some examples of g which satisfy (g1) - (g3) can be seen in [LC18]. We can rewrite the 3D Kelvin-Voigt equations (1.1) in the following functional form

$$\begin{cases} \partial_t(u + \alpha^2 Au) + \nu Au + B(u,u) + \kappa|u|^{\beta-1}u = Pg(t,u_t) + Ph(t), & \text{in } (0,T)\times\Omega \\ u(\theta) = \phi(\theta), & \theta \in (-\infty,0] \end{cases} \#(2.5)$$

We first give the definition of a weak solution.

Definition 2.1. *Given an initial datum $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$, a weak solution u to 1.1 in the interval $(-\infty,T], T > 0$, is a function $u \in C((-\infty,T];H) \cap L^2((0,T);V) \cap L^{\beta+1}((0,T);L^{\beta+1}(\Omega))$ with $u(\theta) = \phi(\theta), \theta \le 0$ and $\frac{du}{dt} \in L^2((0,T);V') + L^{(\beta+1)/\beta}((0,T);L^{(\beta+1)/\beta}(\Omega))$ such that, for all $v \in V$, and a.e. $t \in (0,T)$*

$$\frac{d}{dt}(u(t), v + \alpha^2(u(t),v)) + \nu(u(t),v) + b(u(t),u(t),v) + \langle\kappa|u|^{\beta-1}u, v\rangle$$
$$= \langle h(t), v\rangle + (g(t,u_t), v).$$

The existence and the uniqueness of solution is proved by using the classic Galerkin approximation and the energy method (see [YZ25]).

Theorem 2.1. *Consider $h \in L^2((0,T);V'), g: [0,T] \times BCL_{-\infty}(H) \to H$ satisfying $(g1) - (g3)$ and $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$ are given. Then there exists a unique weak solution to (1.1).*

3. Asymptotical stability of stationary solutions

In order to study the properties of stationary solutions, we need to impose some extra assumptions. Firstly, we assume that h is independent of time, i.e., $h(t) \equiv h \in V'$. Denote by i the trivial immersion $i: H \to BCL_{-\infty}(H)$ given by $i(u) = \tilde{u}$ with $\tilde{u}(t) = u$ for all $t \le 0$. We now require that g satisfies

(g4) $g(s,\xi) = g(t,\xi)$ for any $s,t \in \mathbb{R}_+$ and $\xi \in i(H)$.

If (g2) - (g4) hold, we trivially have that $\tilde{g}: H \to (L^2(\Omega))^3$ defined by $\tilde{g}(u) = g(0,i(u))$, i.e., $\tilde{g} = g_{|\mathbb{R}_+\times i(H)}$, is of course autonomous, Lipschitz (with the same Lipschitz constant L_g) and $\tilde{g}(0) = 0$.

Hence, the stationary equation to (2.5) is the following form which does not contain a delay term:

$$\nu Au + B(u,u) + \kappa|u|^{\beta-1}u = Ph + P\tilde{g}(u). \#(3.1)$$

Let us consider the definition of stationary solutions to problem (1.1).

Definition 3.1. *A weak stationary solution to (1.1) is an element $u^* \in V$ such that*

$$\nu((u^*,v)) + b(u^*,u^*,v) + \langle\kappa|u^*|^{\beta-1}u^*, v\rangle$$
$$= \langle h,v\rangle + (\tilde{g}(u^*),v), \forall v \in V.$$

The existence of stationary solution is established by employing the corollary of the Brouwer fixed point theorem.

Theorem 3.1. *Suppose that the assumptions (g2) - (g4) hold and* $h \in V'$. *If* $2L_g < \nu\lambda_1$ *then problem (1.1) admits at least one stationary solution* u^* *satisfying the following estimate*

$$\|u^*\| \leq \left(\frac{\lambda_1 \|h\|_*^2}{\nu(\lambda_1\nu - 2L_g)}\right)^{\frac{1}{2}}. \#(3.2)$$

Moreover, if the following condition holds

$$\nu > L_g\lambda_1^{-1} + \frac{2c_0\lambda_1^{1/4}\|h\|_*}{\sqrt{\nu(\lambda_1\nu - 2L_g)}} \#(3.3)$$

then the stationary solution of (1.1) is unique.

Definition 3.2. *A stationary* u^* *to (1.1) is stable if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $\phi \in BCL_{-\infty}(H)$ *satisfies* $\|\phi - i(u^*)\|_{BCL_{-\infty}(H)} \leq \delta$, *then the solution* $u(\cdot;\phi)$ *to (1.1) exists for all* $t \geq 0$ *and satisfies* $|u(t;\phi) - u^*| < \varepsilon$ *for any* $t \geq 0$.

We consider the case of $g(t, u_t) = G(u(t - \rho(t)))$, where $G: H \to (L^2(\Omega))^3$ is a measurable function satisfying $G(0) = 0$, and assume that there exists $L_g > 0$ such that

$$|G(u) - G(v)| \leq L_g|u - v|, \forall u, v \in H. \#(3.4)$$

Consider a function $\rho(t) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$with $\rho_* = \max_{t\in[0,T]} \rho'(t) < 1$. The system (2.5) now becomes

$$\frac{d}{dt}(u + \alpha^2 Au) = -\nu Au - B(u, u)$$
$$-P\kappa|u|^{\beta-1}u + Ph + PG(u(t - \rho(t))), \# \qquad (3.5)$$

with initial condition $u(\theta) = \phi(\theta), \theta \in (-\infty, 0]$.

We now show the asymptotical stability of stationary solutions via the construction of Lyapunov functionals.

Theorem 3.2. *Suppose that* $f \in V'$ *and (3.3) hold. If* $\nu\lambda_1 > 2L_g$ *then there exist at least one weak stationary solution* u^* *to (3.1) satisfying (3.2). In addition, if* $\nu \geq 2c_0\lambda_1^{1/4}\left(\frac{\|h\|_*^2}{\nu(\lambda_1\nu - 2L_g)}\right)^{1/2} + \frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}}$

then the stationary solution u^* *is unique, stable and satisfies*

$$\int_0^\infty (|u(s) - u^*|^2 + \alpha^2\|u(s) - u^*\|^2)ds\#$$
$$\leq |\phi(0) - u^*|^2 + \alpha^2\|\phi(0) - u^*\|^2 + \frac{L_g}{\sqrt{1 - \rho_*}}\|\phi - u^*\|^2_{L^2(-\rho(0),0;H)}\# \qquad (3.6)$$

for any solution u *to (3.5) with* $\phi \in BCL_{-\infty}(H)$. *Furthermore, if*

$$\nu > 2c_0\lambda_1^{1/4}\left(\frac{\|h\|_*^2}{\nu(\lambda_1\nu - 2L_g)}\right)^{1/2} + \frac{L_g\lambda_1^{-1}}{\sqrt{1 - \rho_*}}$$

then u^* *is asymptotically stable.*

Proof. Since all the assumptions of Theorem 3.1 are satisfied, then there exists a unique stationary solution u^* to (3.1) satisfying (3.2). Let us set $w(t) = u(t) - u^*$. Then it satisfies

$$\frac{d}{dt}(w(t) + \alpha^2 Aw(t)) = -\nu Aw(t) - B\big(u(t), u(t)\big) + B(u^*, u^*)$$
$$-\kappa|u|^{\beta-1}u + \kappa|u^*|^{\beta-1}u^* + P\left(G\left(u\big(t - \rho(t)\big)\right) - G(u^*)\right) \quad (3.7)$$

with initial condition $w(\theta) = \phi(\theta) - u^*, \theta \in (-\infty, 0]$. For any $\phi \in BCL_{-\infty}(H)$, and any $t > 0$ we define U

$$U(t, \phi) = |\phi(0) - u^*|^2 + \alpha^2\|\phi(0) - u^*\|^2 + \frac{c}{1 - \rho_*}\int_{t-\rho(t)}^t |u(s) - u^*|^2 ds$$

where the constant $c > 0$ is to be chosen later. Then for any $u(.;\phi)$ of 3.5 with initial data $\phi \in BCL_{-\infty}(H)$, we have

$$U(t, u_t) = |u(t) - u^*|^2 + \alpha^2\|u(t) - u^*\|^2 + \frac{c}{1-\rho_*}\int_{t-\rho(t)}^t |u(s) - u^*|^2 ds\# \qquad (3.8)$$

It is well known (see, e.g. [4]) that exists nonnegative constant $\mu = \mu(\beta, \kappa)$ such that

$$\int_\Omega \left(\kappa|u|^{\beta-1}u - \kappa|u^*|^{\beta-1}u^*\right)(u - u^*)dx \geq \int_\Omega \mu\left(|u|^{\beta-1} + |u^*|^{\beta-1}\right)|u - u^*|^2 dx \geq 0$$

From (3.7) and using an estimate similar to 2.4 we obtain

$$\frac{d}{dt}U(t, w_t)$$
$$= 2\left\langle\frac{d}{dt}(w(t) + \alpha^2 Aw(t)), w(t)\right\rangle + \frac{c}{1 - \rho_*}|w(t)|^2 - \frac{c(1 - \rho'(t))}{1 - \rho_*}|w(t - \rho(t))|^2$$
$$\leq -2\nu\|w(t)\|^2 + 4c_0\lambda_1^{-1/4}\|u^*\|\|w(t)\|^2$$
$$+2L_g|w(t - \rho(t))||w(t)| + \frac{c}{1 - \rho_*}|w(t)|^2 - \frac{c(1 - \rho'(t))}{1 - \rho_*}|w(t - \rho(t))|^2.$$

By Cauchy's inequality, Poincaré's inequality (2.1) and (3.2), we obtain

$$\begin{aligned}
&\frac{d}{dt}U(t,w_t)\\
&\le -2\nu\|w(t)\|^2 + 4c_0\lambda_1^{1/4}\left(\frac{\|h\|_*^2}{\nu(\lambda_1\nu - 2L_g)}\right)^{1/2}\|w(t)\|^2\\
&\quad +2L_g|w(t-\rho(t))||w(t)| + \frac{c}{1-\rho_*}|w(t)|^2 - c|w(t-\rho(t))|^2\\
&\le -2\nu\|w(t)\|^2 + 4c_0\lambda_1^{1/4}\left(\frac{\|h\|_*^2}{\nu(\lambda_1\nu - 2L_g)}\right)^{1/2}\|w(t)\|^2\\
&\quad +2\left(\frac{c}{2}|w(t-\rho(t))|^2 + \frac{L_g^2}{2c}|w(t)|^2\right) + \frac{c}{1-\rho_*}|w(t)|^2 - c|w(t-\rho(t))|^2\\
&\le -2\left(\nu - 2c_0\lambda_1^{1/4}\left(\frac{\|h\|_*^2}{\nu(\lambda_1\nu - 2L_g)}\right)^{1/2} - \frac{L_g^2\lambda_1^{-1}}{2c} - \frac{c\lambda_1^{-1}}{2(1-\rho_*)}\right)\times\|w(t)\|^2
\end{aligned}$$

If we choose $c = L_g\sqrt{1-\rho_*}$, then the coefficient in the right-hand side takes it minimum value. We conclude that

$$\frac{d}{dt}U(t,w_t) \le -2\left(\nu - 2c_0\lambda_1^{\frac{1}{4}}\left(\frac{\|h\|_*^2}{\nu(\lambda_1\nu - 2L_g)}\right)^{\frac{1}{2}} - \frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}}\right)\|w(t)\|^2. \quad (3.9)$$

Integrating (3.9) from 0 to t, we obtain

$$\begin{aligned}
U(t,w_t) &+ 2\left(\nu - 2c_0\lambda_1^{1/4}\left(\frac{\|h\|_*^2}{\nu(\lambda_1\nu - 2L_g)}\right)^{1/2} - \frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}}\right)\int_0^t \|w(s)\|^2 ds\\
&\le U(0,u_0) \quad (3.10)
\end{aligned}$$

From (3.8), we have

$$U(t,w_t) \ge |u(t)-u^*|^2 + \alpha^2\|u(t)-u^*\|^2$$

and

$$U(0,u_0) = |\phi(0)-u^*|^2 + \alpha^2\|\phi(0)-u^*\|^2 + \frac{L_g}{\sqrt{1-\rho_*}}\|\phi - u^*\|^2_{L^2(-\rho(0),0;H)}$$

Then using the Poincaré inequality (2.1), (3.10) becomes

$$\begin{aligned}
&|u(t)-u^*|^2 + \alpha^2\|u(t)-u^*\|^2\\
&+2\lambda_1\left(\nu - 2c_0\lambda_1^{\frac{1}{4}}\left(\frac{\|h\|_*^2}{\nu(\lambda_1\nu - 2L_g)}\right)^{\frac{1}{2}} - \frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}}\right)\times\int_0^t |u(s)-u^*|^2 ds\\
&\le |\phi(0)-u^*|^2 + \alpha^2\|\phi(0)-u^*\|^2 + \frac{L_g}{\sqrt{1-\rho_*}}\|\phi-u^*\|^2_{L^2(-\rho(0),0;H)}. \quad (3.11)
\end{aligned}$$

Therefore, if $\nu \ge 2c_0\lambda_1^{1/4}\left(\frac{\|h\|_*^2}{\nu(\lambda_1\nu-2L_g)}\right)^{1/2} + \frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}}$ then the stationary solution u^* is stable and satisfies (3.6).

If $\nu > 2c_0\lambda_1^{1/4}\left(\frac{\|h\|_*^2}{\nu(\lambda_1\nu-2L_g)}\right)^{1/2} + \frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}}$, from (3.11) we obtain

$$\begin{aligned}
&\int_0^\infty (|u(s)-u^*|^2 + \alpha^2\|u(s)-u^*\|^2)ds\\
&\le |\phi(0)-u^*|^2 + \alpha^2\|\phi(0)-u^*\|^2 + \frac{L_g}{\sqrt{1-\rho_*}}\|\phi-u^*\|^2_{L^2(-\rho(0),0;H)}
\end{aligned}$$

By the continuity in time of u in H, we deduce that $\lim_{t\to\infty}|u(t)-u^*|^2 = 0$, i.e. the stationary solution u^* is asymptotically stable.

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